THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Advanced Calculus I
Tutorial 12

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1. Let $G(x, y) = x^2 - 3xy + y^3 - 7$. Show that $(4, 3)$ is a solution and show that the function implicitly defines y as a function of x, i.e., $y = y(x)$ near $(4, 3)$. Moreover, find a formula for $\dfrac{\partial y}{\partial x}$ in terms of x and y and evaluate it at $(4, 3)$.

2. Let S be the level surface given by
$$x^3 + 3xyz + z^2 y - 4y - 5z + 4 = 0, (x, y, z) \in \mathbb{R}^3.$$
Determine which variables can be written as functions of the other two near $(1, 1, 1)$ for this system. Find the equation of the tangent plane to S at $(1, 1, 1)$.

3. Consider the system
$$\begin{cases} F(x, y, u, v) = xye^u + \sin(v - u) & = 0 \\ G(x, y, u, v) = (x + 1)(y + 2)(u + 3)(v + 4) - 24 & = 0 \end{cases}$$
and note that $(0, 0, 0, 0)$ is a solution. Does the system of equations implicitly determine (u, v) as a function of (x, y), i.e., $(u, v) = \vec{f}(x, y)$, for (x, y) near $(0, 0)$? If so, find $\nabla_x \vec{f}(x, y)$ at $(0, 0)$.

4. Consider the system
$$\begin{cases} F(x, y, z) = x^2 y + 3y + z^3 - z & = 8 \\ G(x, y, z) = 2x + 2y + \cos(xz) & = 7 \end{cases}$$
Determine, for any pair of variables $(x, y), (x, z)$ or (y, z), whether it is possible to solve for this system for those two variables as C^1 functions of the third variable near the point $(x, y, z) = (1, 2, 0)$. If (y, z) can be written as functions of x, say $\binom{y}{z} = \binom{f_1(x)}{f_2(x)}$ for x close to 1, then determine $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ at 1.

1. Let $G(x, y) = x^2 - 3xy + y^3 - 7$. Show that $(4, 3)$ is a solution and show that the function implicitly defines y as a function of x, i.e., $y = y(x)$ near $(4, 3)$. Moreover, find a formula for $\frac{\partial y}{\partial x}$ in terms of x and y and evaluate it at $(4, 3)$.

$$G(4,3) = 4^2 - 3\cdot 4\cdot 3 + 3^3 - 7 = 0.$$

$$\left.\frac{\partial G}{\partial y}\right|_{(4,3)} = -3x + 3y^2\Big|_{(4,3)} = -12 + 3\cdot 3^2 = 15 \neq 0.$$

Impl FT $\Rightarrow$ $y = y(x)$ near $(4,3)$.

Implicitly differentiate G wrt. x:

$$2x - 3y\frac{\partial y}{\partial x} + 3y^2\frac{\partial y}{\partial x} = 0.$$

$$\Rightarrow \left.\frac{\partial y}{\partial x}\right|_{(4,3)} = \left.\frac{-2x}{-3y+3y^2}\right|_{(4,3)} = -\frac{1}{6}.$$

//.

2. Let S be the level surface given by

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Determine which variables can be written as functions of the other two near $(1, 1, 1)$ for this system. Find the equation of the tangent plane to S at $(1, 1, 1)$.

let $F(x,y,z) = x^3 + 3xyz + z^2y - 4y - 5z + 4$.

Then $F(1,1,1) = 0$.

$$\left.\frac{\partial F}{\partial x}\right|_{(1,1,1)} = 3x^2 + 3yz\Big|_{(1,1,1)} = 6 \neq 0$$

Impl. F.T. $\Rightarrow$ $x = x(y,z)$ near $(1,1,1)$.

$$\left.\frac{\partial F}{\partial y}\right|_{(1,1,1)} = 3xz + z^2 - 4\Big|_{(1,1,1)} = 0 \quad \times$$

$$\left.\frac{\partial F}{\partial z}\right|_{(1,1,1)} = 3xy + 2zy - 5\Big|_{(1,1,1)} = 0 \quad \times.$$

Equ. of Tangent Plane:

$$\left.\frac{\partial F}{\partial x}\right|_{(1,1,1)}(x-1) + \cancel{\left.\frac{\partial F}{\partial y}\right|_{(1,1,1)}}^{\,0}(y-1) + \cancel{\left.\frac{\partial F}{\partial z}\right|_{(1,1,1)}}^{\,0}(z-1) = 0$$

$6x = 6 \Rightarrow x = 1$. //

3. Consider the system

$$(*)\quad \begin{cases} F(x,y,u,v) = xye^u + \sin(v-u) & = 0 \\ G(x,y,u,v) = (x+1)(y+2)(u+3)(v+4) - 24 & = 0 \end{cases}$$

and note that $(0,0,0,0)$ is a solution. Does the system of equations implicitly determine (u,v) as a function of (x,y), i.e., $(u,v) = \vec{f}(x,y)$, for (x,y) near $(0,0)$? If so, find $\nabla_x \vec{f}(x,y)$ at $(0,0)$.

$$\begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{pmatrix} = \begin{pmatrix} xye^u - \cos(v-u) & \cos(v-u) \\ (x+1)(y+2)(v+4) & (x+1)(y+2)(u+3) \end{pmatrix}$$

at $(0,0,0,0)$: $= \begin{pmatrix} -1 & 1 \\ 8 & 6 \end{pmatrix}$

So $\det\begin{pmatrix} -1 & 1 \\ 8 & 6 \end{pmatrix} = -6 - 8 = -14 \neq 0$. so invertible

Impl. F.T.

$\Rightarrow$ can write $(u,v) = (f_1(x,y), f_2(x,y)) = \vec{f}(x,y)$

By implicit diff. of $(*)$ wrt. x (y indep. of x, so $\frac{\partial y}{\partial x} = 0$)

$$\begin{cases} ye^u + xye^u \frac{\partial u}{\partial x} - \cos(v-u)\frac{\partial u}{\partial x} + \cos(v-u)\frac{\partial v}{\partial x} = 0 \\ (y+2)(u+3)(v+4) + (x+1)(u+3)(v+4)\cancelto{0}{\frac{\partial y}{\partial x}} \\ \quad + (x+1)(y+2)(v+4)\frac{\partial u}{\partial x} + (x+1)(y+2)(u+3)\frac{\partial v}{\partial x} = 0 \end{cases}$$

Evaluate at $(0,0,0,0)$, we get:

$$\begin{cases} 0 + 0\cdot\frac{\partial u}{\partial x} - 1\cdot\frac{\partial u}{\partial x} + 1\cdot\frac{\partial v}{\partial x} = 0 \\ 24 + 8\frac{\partial u}{\partial x} + 6\frac{\partial v}{\partial x} = 0. \end{cases} \Rightarrow \begin{pmatrix} -1 & 1 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ -24 \end{pmatrix}$$

So $\nabla_x \vec{f}(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 8 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -24 \end{pmatrix}$

$= -\frac{1}{14} \begin{pmatrix} 6 & -1 \\ -8 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -24 \end{pmatrix} = \begin{pmatrix} -12/7 \\ -12/7 \end{pmatrix}$

4. Consider the system

$$(*)\ \begin{cases} F(x,y,z) = x^2y + 3y + z^3 - z & = 8 \\ G(x,y,z) = 2x + 2y + \cos(xz) & = 7 \end{cases}$$

Determine, for any pair of variables $(x,y),(x,z)$ or (y,z), whether it is possible to solve for this system for those two variables as C^1 functions of the third variable near the point $(x,y,z) = (1,2,0)$. If (y,z) can be written as functions of x, say $\binom{y}{z} = \binom{f_1(x)}{f_2(x)}$ for x close to 1, then determine $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ at 1.

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{pmatrix} = \begin{pmatrix} 2xy & x^2+3 & 3z^2-1 \\ 2-z\sin(xz) & 2 & -x\sin(xz) \end{pmatrix}$$

at $(1,2,0)$:

$$= \begin{array}{c} \begin{pmatrix} 4 & 4 & -1 \\ 2 & 2 & 0 \end{pmatrix} \\ \begin{matrix} x & \ \ y & \ \ z \end{matrix} \end{array}$$

xy: $\begin{vmatrix} 4 & 4 \\ 2 & 2 \end{vmatrix} = 4\cdot 2 - 4\cdot 2 = 0$. ✗.

xz: $\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix} = 4\cdot 0 - (-1)\cdot 2 = 2 \neq 0$. ✓.

yz: $\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix} = 2 \neq 0$. ✓.

So $(x,z) = (g_1(y), g_2(y))$ and $(y,z) = (f_1(x), f_2(x))$ near $(1,2,0)$.

Implicitly differentiate (*) wrt. x:

$$\begin{cases} 2xy + x^2\dfrac{\partial y}{\partial x} + 3\dfrac{\partial y}{\partial x} - 3z^2\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial x} = 0 \\[2ex] 2x + 2\dfrac{\partial y}{\partial x} - z\sin(xz) - x\sin(xz)\dfrac{\partial z}{\partial x} = 0 \end{cases}$$

at $(1,2,0)$:

$$\begin{cases} 4\dfrac{\partial y}{\partial x} - \dfrac{\partial z}{\partial x} = -4 \\[2ex] 2\dfrac{\partial y}{\partial x} - 0\cdot\dfrac{\partial z}{\partial x} = -2 \end{cases} \Rightarrow \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \dfrac{dy}{dx} \\[1ex] \dfrac{dz}{dx} \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dfrac{dy}{dx} \\[1ex] \dfrac{dz}{dx} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}^{-1}\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$